

OPERATOR SPLITTING FOR NONAUTONOMOUS DELAY EQUATIONS

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ABSTRACT. We provide a general product formula for the solution of nonautonomous abstract delay equations. After having shown the convergence we obtain estimates on the order of convergence for differentiable history functions. Finally, the theoretical results are demonstrated on some typical numerical examples.

1. INTRODUCTION

Operator splitting is a widely used time discretization method for the numerical solution of complicated equations. The importance and main applications of these procedures is described, for example, in the monographs by Faragó and Havasi [7], Holden et al. [9] and Lubich [10].

The present paper investigates a special operator splitting for a class of nonautonomous delay differential equations. This method, which can be applied to equations with distributed delays very effectively, was first investigated in Csomós and Nickel [5] and in Bátkai, Csomós and Nickel [2] in the autonomous case. Recall from Bellen and Zennaro [4] that delay equations with distributed delay, especially those where the delay term is not separated from zero, are particularly difficult to solve numerically. Nevertheless, as we shall see, splitting methods work quiet well even in the latter case.

To motivate this approach, let us consider the following equation.

$$\begin{cases} \dot{u}(t) = b(t)u(t) + \int_{-1}^0 \mu(t, \sigma)u(t + \sigma)d\sigma, & t \geq s, \\ u(s) = x \in \mathbb{R}, \\ u(s + \sigma) = f(\sigma), & \sigma \in [-1, 0], \end{cases}$$

where $b \in C_b^1(\mathbb{R})$, $\mu \in L^\infty(\mathbb{R} \times [-1, 0])$, and $t \mapsto \mu(t, \sigma) \in C_b^1(\mathbb{R})$ for all $\sigma \in [-1, 0]$. In this case the delay operator $\Phi(t)$ is defined by

$$\Phi(t)g := \int_{-1}^0 \mu(t, \sigma)g(\sigma)d\sigma$$

for all $g \in L^1([-1, 0])$.

Choosing a time step $h \in (0, 1]$, first we start with $x_0 := x$ and $f_0 := f$. Then we set

$$x_1 := e^{hb(s)}(x_0 + h\Phi(s)f_0)$$

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and

$$f_1(\sigma) := \begin{cases} e^{(h+\sigma)b(s)}(x_0 + h\Phi(s)f_0), & \sigma \in [-h, 0], \\ f_0(h + \sigma), & \sigma \in [-1, -h]. \end{cases}$$

In the next step we repeat this procedure and replace x_0 with x_1 , f_0 with f_1 , and s with $s + h$. Hence, we obtain an iteration process where in the k^{th} step we have

$$(1) \quad \begin{cases} x_k := e^{hb(s+h(k-1))}(x_{k-1} + h\Phi(s+h(k-1))f_{k-1}), \\ f_k(\sigma) := \begin{cases} e^{(h+\sigma)b(s+h(k-1))}(x_{k-1} + h\Phi(s+h(k-1))f_{k-1}), & \sigma \in [-h, 0], \\ f_{k-1}(h + \sigma), & \sigma \in [-1, -h]. \end{cases} \end{cases}$$

The aim of the present paper is to explain the convergence of this procedure also for more general equations. We do this by introducing an abstract setup allowing us a general convergence result of the procedure. Further, for differentiable initial function (i.e., classical solutions) we also obtain estimates on the order of convergence. In the following, we summarize some basic facts on product formulae for abstract evolution equations. Then in Section 2 we rewrite the nonautonomous delay equation as an abstract evolution equation and prove the convergence of a general product formula. Section 3 is devoted to the investigation of the order of convergence, and in Section 4 we present numerical examples demonstrating the power of this approach.

First, let us recall some general facts about splitting of nonautonomous equations. Consider an evolution equation of the form

$$(NCP) \quad \begin{cases} \frac{d}{dt}u(t) = (A(t) + B(t))u(t), & t \geq s \in \mathbb{R}, \\ u(s) = y \in X. \end{cases}$$

We always suppose that this equation (NCP) is well-posed, i.e., there is an evolution family W (also called semi-dynamical system) solving it. For well-posedness of nonautonomous evolution equations we refer to the surveys in Nagel and Nickel [11], Pazy [12] or Schnaubelt in [6].

Now, a splitting formula, as below, is especially useful, if we are able to solve effectively the *autonomous* Cauchy problems

$$\begin{aligned} \frac{d}{dt}u(t) &= A(r)u(t) \\ \frac{d}{dt}v(t) &= B(r)v(t) \end{aligned}$$

with appropriate initial conditions for every fixed r . This is usually the case, if the operators $A(r)$ and $B(r)$ are partial differential operators with time dependent coefficients, or time dependent multiplication operators, and this is particularly so for delay equations considered in this paper.

The following general convergence result can be proved, see Bátkai et al. [1, Theorem 4.2].

Theorem 1.1. *Suppose the following:*

- a) *The nonautonomous Cauchy problem corresponding to the operators $(A(\cdot) + B(\cdot))$ is well-posed. We denote the evolution family solving (NCP) by W .*
- b) *The operators $A(t)$ and $B(t)$ are generators of C_0 -semigroups of type (M, ω) ($M \geq 1$ and $\omega \in \mathbb{R}$), and*

$$(\omega, \infty) \subset \rho(A(t)) \cap \rho(B(t)) \quad \text{for all } t \in \mathbb{R}$$

and

$$\sup_{s \in \mathbb{R}} \left\| \prod_{p=n}^1 \left(e^{\frac{t}{n}A(s-p\frac{t}{n})} e^{\frac{t}{n}B(s-p\frac{t}{n})} \right) \right\| \leq M e^{\omega t}.$$

c) The maps

$$t \mapsto R(\lambda, A(t))y, \quad t \mapsto R(\lambda, B(t))y$$

are continuous for all $\lambda > \omega$ and $y \in X$.

Then one has the convergence

$$(2) \quad W(t, s)y = \lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} \left(e^{\frac{t-s}{n} A(s+i\frac{t-s}{n})} e^{\frac{t-s}{n} B(s+i\frac{t-s}{n})} \right) y$$

for all $y \in X$, locally uniformly in s, t with $s \leq t$.

2. SPLITTING FOR THE DELAY EQUATION

Consider the **abstract delay equation** in the following form:

$$(3) \quad \begin{cases} \frac{d}{dt}u(t) = A(t)u(t) + \Phi(t)u_t, & t \geq s, \\ u(s) = x \in X, \quad s \in \mathbb{R}, \\ u_s = f \in L^1([-1, 0]; X) \end{cases}$$

on the Banach space X , where $A(t)$ generates a strongly continuous contraction semigroup on X and $\Phi(t) : L^1([-1, 0]; X) \rightarrow X$ is a bounded and linear operator depending continuously on the parameter $t \in \mathbb{R}$. The **history function** u_t is defined by $u_t(\sigma) := u(t + \sigma)$ for $\sigma \in [-1, 0]$. Note that point delays are excluded from this context, but *distributed delays*, even those that live up to 0, are contained in this setting.

In order to rewrite (3) as an abstract Cauchy problem, we take the product space $\mathcal{E} := X \times L^1([-1, 0]; X)$ equipped with 1-sum norm, and the new unknown function as

$$t \mapsto \mathcal{U}(t) := \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{E}.$$

Then (3) can be written as an abstract Cauchy problem on the space \mathcal{E} in the following way:

$$(4) \quad \begin{cases} \frac{d}{dt}\mathcal{U}(t) = \mathcal{G}(t)\mathcal{U}(t), & t \geq s, \\ \mathcal{U}(s) = \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{E}, \end{cases}$$

where the operator $\mathcal{G}(t)$ is given by the matrix

$$(5) \quad \mathcal{G}(t) := \begin{pmatrix} A(t) & \Phi(t) \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

on the domain

$$D(\mathcal{G}(t)) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A(t)) \times W^{1,1}([-1, 0]; X) : f(0) = x \right\}.$$

As in Bátkai and Piazzera [3, Corollary 3.5, Proposition 3.9] one can show that the delay equation (3) and the abstract Cauchy problem (4) are equivalent, i.e., they have the same solutions. More precisely, the first coordinate of the solution of (4) always solves (3), i.e.,

$$u(t) = \pi_1 \mathcal{U}(t),$$

where π_1 is the projection to the first coordinate in \mathcal{E} . Due to this equivalence, the delay equation is well-posed if and only if the operator $\mathcal{G}(t)$ generates an evolution family on the space \mathcal{E} .

Since the delay operators $\Phi(t)$ are bounded, the delay equation (3) is well-posed, which follows from a much more general well-posedness result by Hadd, Rhoads and Schnaubelt [8, Proposition 3.5]. That is, there is an *evolution family* \mathcal{W} such

that for fixed s and $t \geq s$ the function $u^{(s)}(t) = \pi_1 \mathcal{W}(t, s) \binom{x}{f}$ is a solution of (3) for $\binom{x}{f} \in D(\mathcal{G}(s))$. In particular, for $0 \leq s \leq t$ we have

$$\mathcal{W}(t, s) \binom{x}{f} = \begin{pmatrix} u^{(s)}(t) \\ u_t^{(s)} \end{pmatrix},$$

$$\text{where } u^{(s)} \text{ fulfills } u^{(s)}(t) = x + \int_s^t A(r) u^{(s)}(r) dr + \int_s^t \Phi(r) u_r^{(s)} dr$$

$$\text{and } u_t^{(s)}(r) = u^{(s)}(t+r) \quad \text{for } r \in [-1, 0], t+r \geq s.$$

Furthermore, we have the next relation

$$u_t^{(s)}(r) = f(r+t-s) \quad \text{for } t+r < s.$$

Now we make the main assumptions implying the convergence of the splitting procedure. In the autonomous case, i.e., when $A(t) = A$ and $\Phi(t) = \Phi$, the following was investigated in the papers by Csomós and Nickel [5] and Bátkai, Csomós, and Nickel [2].

- Assumption 2.1.** a) The operators $A(s)$ generate the strongly continuous contraction semigroups $(V^{(s)}(t))_{t \geq 0}$ on X for all $s \in \mathbb{R}$.
b) $D(A(s)) =: D$ for all $s \in \mathbb{R}$ and the function $s \mapsto R(1, A(s))$ is continuous.
c) The delay operators $\Phi(s) : L^1([-1, 0]; X) \rightarrow X$ are bounded for all $s \in \mathbb{R}$.
d) The function $s \mapsto \Phi(s)f$ is bounded and continuous for every $f \in L^1([-1, 0]; X)$.

Let us now describe in detail the approximation procedure we will apply. We split the operator in (4) as

$$\mathcal{G}(t) = \mathcal{A}(t) + \mathcal{B}(t),$$

where the sub-operators have the form

$$(6) \quad \begin{aligned} \mathcal{A}(r) &:= \begin{pmatrix} A(r) & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, & D(\mathcal{A}(r)) &:= D(\mathcal{G}(r)), \\ \mathcal{B}(r) &:= \begin{pmatrix} 0 & \Phi(r) \\ 0 & 0 \end{pmatrix}, & D(\mathcal{B}(r)) &:= \mathcal{E}. \end{aligned}$$

Since $A(r)$ is a generator and $\Phi(r)$ is bounded, the operators $\mathcal{A}(r)$ and $\mathcal{B}(r)$ generate the strongly continuous semigroups $(\mathcal{T}^{(r)}(t))_{t \geq 0}$ and $(\mathcal{S}^{(r)}(t))_{t \geq 0}$, respectively. It is shown in Bátkai and Piazzera [3, Theorem 3.25] that \mathcal{T} is given by

$$\mathcal{T}^{(r)}(t) := \begin{pmatrix} V^{(r)}(t) & 0 \\ V_t^{(r)} & T(t) \end{pmatrix},$$

where $(T(t))_{t \geq 0}$ is the nilpotent left shift semigroup defined by

$$(T(t)f)(\sigma) := \begin{cases} f(t+\sigma), & \text{if } \sigma \in [-1, -t), \\ 0, & \text{if } \sigma \in [-t, 0] \end{cases}$$

for all $f \in L^1([-1, 0]; X)$, and $V_t^{(r)}$ is

$$(V_t^{(r)}x)(\sigma) := \begin{cases} V^{(r)}(t+\sigma)x, & \text{if } \sigma \in [-t, 0], \\ 0, & \text{if } \sigma \in [-1, -t) \end{cases}$$

for all $x \in X$. Since $\Phi(r)$ is a bounded operator, $\mathcal{B}(r)$ is also bounded on \mathcal{E} . Therefore, the semigroup $\mathcal{S}(r)$ generated by $\mathcal{B}(r)$ takes the form

$$\mathcal{S}(r)(t) := e^{t\mathcal{B}(r)} = \mathcal{I} + t\mathcal{B}(r) = \begin{pmatrix} I & t\Phi(r) \\ 0 & \tilde{I} \end{pmatrix},$$

where I , \tilde{I} , and \mathcal{I} denote the identity operators on X , $L^1([-1, 0]; X)$, and \mathcal{E} , respectively. We then have the following general convergence result explaining the convergence of the procedure described in (1).

Theorem 2.2. *Under Assumption 2.1 the solution of the abstract delay equation (3) is given by the formula*

$$u^{(s)}(t) = \pi_1 \lim_{n \rightarrow \infty} \prod_{p=0}^{n-1} \begin{pmatrix} V^{(s+p\frac{t-s}{n})}(\frac{t-s}{n}) & 0 \\ V^{(s+p\frac{t-s}{n})}(\frac{t-s}{n}) & T(\frac{t-s}{n}) \end{pmatrix} \begin{pmatrix} I & \frac{t-s}{n} \Phi(s + p\frac{t-s}{n}) \\ 0 & \tilde{I} \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix}.$$

Proof. The convergence is a direct consequence of Theorem 1.1 applied to the generators $\mathcal{A}(r)$ and $\mathcal{B}(r)$. As mentioned above, by Hadd, Rhandi and Schnaubelt [8] the Cauchy problem associated to the operator $\mathcal{A}(r) + \mathcal{B}(t)$ is well-posed. Applying Theorem 1.1, we only have to check the stability assumption. By using the same arguments as the ones appearing in the proof of Csomós and Nickel [5, Theorem 4.2], we get that

$$\begin{aligned} \|\mathcal{T}^{(r)}(t) \begin{pmatrix} x \\ f \end{pmatrix}\|_{\mathcal{E}} &= \|V^{(r)}(t)x\| + \|V_t^{(r)}x + T(t)f\|_1 \leq \|x\| + \|f\|_1 + t\|x\| \\ &\leq (1+t)(\|x\| + \|f\|_1) = (1+t)\left\|\begin{pmatrix} x \\ f \end{pmatrix}\right\|_1, \end{aligned}$$

hence

$$\|\mathcal{T}^{(r)}(t) \begin{pmatrix} x \\ f \end{pmatrix}\|_{\mathcal{E}} \leq 1 + t.$$

One also obtains

$$\|\mathcal{S}^{(r)}(t)\| \leq 1 + t\|\Phi(r)\|.$$

From this the stability can be obtained:

$$\begin{aligned} \left\| \prod_{i=n}^1 \begin{pmatrix} V^{(s-i\frac{t}{n})}(\frac{t}{n}) & 0 \\ V^{(s-i\frac{t}{n})}(\frac{t}{n}) & T(\frac{t}{n}) \end{pmatrix} \begin{pmatrix} I & \frac{t}{n} \Phi(s - i\frac{t}{n}) \\ 0 & \tilde{I} \end{pmatrix} \right\| &\leq \left(1 + \frac{t}{n} \sup_{s \in \mathbb{R}} \|\Phi(s)\|\right)^n \left(1 + \frac{t}{n}\right)^n \\ &\leq e^{\omega t}, \end{aligned}$$

with $\omega = 1 + \sup_{s \in \mathbb{R}} \|\Phi(s)\|$. □

3. ORDER OF CONVERGENCE

We now investigate the order of convergence of the splitting method from the previous section. To this end, we have to make further assumptions on the operators involved, and of course, some regularity on the initial data need to be assumed, too.

Assumption 3.1.

- a) The operator $A(s)$ is bounded and generates the strongly continuous contraction semigroup $(V^{(s)}(t))_{t \geq 0}$ on X .
- b) The delay operators $\Phi(s) : L^1([-1, 0]; X) \rightarrow X$ are bounded for all $s \in \mathbb{R}$.
- c) The function $s \mapsto A(s)x$ is bounded and locally Lipschitz continuous, i.e., for all $T_0 > 0$ there is $L_{T_0} \geq 0$ such that

$$\|A(s)x - A(t)x\| \leq L_{T_0}\|x\||t - s|$$

for all $|t|, |s| \leq T_0$.

- d) The function $s \mapsto \Phi(s)f$ is bounded and locally Lipschitz continuous, i.e., for all $T_0 > 0$ there is $L_{T_0} \geq 0$ such that

$$\|\Phi(s)f - \Phi(t)f\| \leq L_{T_0}\|f\|_1|t - s|$$

for all $|t|, |s| \leq T_0$.

These assumptions enable us to show the first order of convergence for classical solutions.

Theorem 3.2 (Local error estimate). *Let $T_0 > 0$ be fixed. Then there is a constant $C > 0$ such that*

$$\|\mathcal{T}^{(s)}(h)\mathcal{S}^{(s)}(h)\binom{x}{f} - \mathcal{W}(s+h, s)\binom{x}{f}\| \leq Ch^2(\|x\| + \|f\|_1 + \|f'\|_1)$$

holds for all $h \in [0, 1]$, $s \in [-T_0, T_0]$ and $f \in W^{1,1}([-1, 0]; X)$, $x = f(0)$.

Proof. Recall from the previous section that for $0 \leq s \leq s+h$ we have

$$\mathcal{W}(s+h, s)\binom{x}{f} = \begin{pmatrix} u^{(s)}(s+h) \\ u_{s+h}^{(s)} \end{pmatrix},$$

$$\text{where } u^{(s)} \text{ fulfills } u^{(s)}(s+h) = x + \int_s^{s+h} A(r)u^{(s)}(r)dr + \int_s^{s+h} \Phi(r)u_r^{(s)}dr$$

$$\text{and } u_h^{(s)}(r) = u^{(s)}(h+r) \text{ for } h+r \geq s.$$

From this it follows that $u^{(s)} : [s, s+1] \rightarrow X$ is Lipschitz continuous with constant $L(\|x\| + \|f\|_1)$ with L dependent only on $\|A\|_\infty$ and $\|\Phi\|_\infty$. Let us calculate the product

$$(7) \quad \mathcal{T}^{(s)}(h)\mathcal{S}^{(s)}(h) = \begin{pmatrix} V^{(s)}(h)x + hV^{(s)}(h)\Phi(s)f \\ V_h^{(s)}x + hV_h^{(s)}\Phi(s)f + T(h)f \end{pmatrix},$$

and compare the first component here with $u^{(s)}(s+h)$. We can write

$$\begin{aligned} & V^{(s)}(h)x + hV^{(s)}(h)\Phi(s)f - u^{(s)}(s+h) \\ &= V^{(s)}(h)x + hV^{(s)}(h)\Phi(s)f - x - \int_s^{s+h} A(r)u^{(s)}(r)dr - \int_s^{s+h} \Phi(r)u_r^{(s)}dr, \end{aligned}$$

and by writing out the series expansion of $V^{(s)}(h)$ we obtain

$$\begin{aligned} & V^{(s)}(h)x + hV^{(s)}(h)\Phi(s)f - u^{(s)}(s+h) \\ &= x + hA(s)x + h\Phi(s)f + O(h^2) - x - \int_s^{s+h} A(r)u^{(s)}(r)dr - \int_s^{s+h} \Phi(r)u_r^{(s)}dr \\ (8) \quad &= hA(s)x - \int_s^{s+h} A(r)u^{(s)}(r)dr + \Phi(s)f - \int_s^{s+h} \Phi(r)u_r^{(s)}dr + O(h^2), \end{aligned}$$

where $O(h^2)$ denotes a term bounded in norm by $C \cdot h^2(\|x\| + \|f\|_1)$ with a constant C that depends only on the bounds of $\|A\|_\infty$ and $\|\Phi\|_\infty$. We now can write

$$\begin{aligned} & \left\| hA(s)x - \int_s^{s+h} A(r)u^{(s)}(r)dr \right\| \leq \int_s^{s+h} \|A(s)x - A(r)u^{(s)}(r)\|dr \\ & \leq \int_s^{s+h} \|A(s)x - A(r)x\|dr + \int_s^{s+h} \|A(r)x - A(r)u^{(s)}(r)\|dr \\ & \leq (L'\|x\| + \|A\|_\infty L) \int_s^{s+h} (r-s)dr = O(h^2), \end{aligned}$$

where L' is the Lipschitz constant of A on $[-T_0, T_0 + 1]$. A very similar reasoning works for the other two terms in (8):

$$\begin{aligned}
\left\| t\Phi(s)f - \int_s^{s+h} \Phi(r)u_r^{(s)} \right\| &\leq \int_s^{s+h} \|\Phi(s)f - \Phi(r)u_r^{(s)}\| dr \\
&\leq \int_s^{s+h} \|\Phi(s)f - \Phi(r)f\| dr + \int_s^{s+h} \|\Phi(r)f - \Phi(r)u_r^{(s)}\| dr \\
&\leq L'\|f\|_1 \int_s^{s+h} (r-s) dr + \|\Phi\|_\infty \int_s^{s+h} \int_{-1}^0 \|f(\sigma) - u_r^{(s)}(\sigma)\| d\sigma dr \\
&= \frac{L'}{2}\|f\|_1 h^2 + \|\Phi\|_\infty \int_s^{s+h} \int_{-1}^{s-r} \|f(\sigma) - f(\sigma+r-s)\| d\sigma dr \\
&\quad + \|\Phi\|_\infty \int_s^{s+h} \int_{s-r}^0 \|f(\sigma) - u^{(s)}(\sigma+r)\| d\sigma dr \\
&= \frac{L'}{2}\|f\|_1 h^2 + \|\Phi\|_\infty \int_s^{s+h} \int_{-1}^{s-r} \|f(\sigma) - f(\sigma+r-s)\| d\sigma dr \\
&\quad + \|\Phi\|_\infty \max\{\|f\|_\infty, \|u^{(s)}|_{[s,s+1]}\|_\infty\} \frac{h^2}{2}.
\end{aligned}$$

Since $f \in W^{1,1}([-1, 0]; X)$, we can continue the estimation:

$$\begin{aligned}
\left\| t\Phi(s)f - \int_s^{s+h} \Phi(r)u_r^{(s)} \right\| &\leq \frac{L'}{2}\|f\|_1 h^2 + \|\Phi\|_\infty \int_s^{s+h} \|f'\|_1 (r-s) dr \\
&\quad + \|\Phi\|_\infty \max\{\|f\|_\infty, \|u^{(s)}|_{[s,s+1]}\|_\infty\} \frac{h^2}{2} \\
&\leq C(\|x\| + \|f\|_1 + \|f'\|_1) h^2.
\end{aligned}$$

By summing up we obtain the estimate:

$$\|V^{(s)}(h)x + hV^{(s)}(h)\Phi(s)f - u^{(s)}(h+s)\| \leq C(\|x\| + \|f\|_1 + \|f'\|_1) h^2.$$

Hence the assertion for the first coordinate in (7) is proved.

Let us now turn our attention to the second coordinate of (7). For $r \in [-1, 0]$ we have the following:

If $h+r \geq 0$

$$\begin{aligned}
&(V_h^{(s)}x + hV_h^{(s)}\Phi(s)f + T(h)f - u_{s+h}^{(s)})(r) \\
&= V^{(s)}(h+r)(x + h\Phi(s)f) + 0 - u^{(s)}(s+h+r),
\end{aligned}$$

and if $h+r < 0$

$$\begin{aligned}
&(V_h^{(s)}x + hV_h^{(s)}\Phi(s)f + T(h)f - u_{s+h}^{(s)})(r) \\
&= 0 - 0 + f(h+r-s) - f(h+r-s) = 0.
\end{aligned}$$

We estimate the L^1 -norm of

$$V_h^{(s)}x + hV_h^{(s)}\Phi(s)f + T(h)f - u_h^{(s)}.$$

By using what is proved in the above for the first coordinate, the pointwise estimate for the integrand, we obtain that

$$\begin{aligned}
& \|V_h^{(s)}x + hV_h^{(s)}\Phi f + T(h)f - u_{s+h}^{(s)}\|_1 \\
&= \int_{-h}^0 \|V^{(s)}(h+r)x + hV^{(s)}(h+r)\Phi(s)f - u^{(s)}(s+h+r)\|dr \\
&\leq \int_{-h}^0 \|V^{(s)}(h+r)x + (h+r)V^{(s)}(h+r)\Phi(s)f - u^{(s)}(s+h+r)\|dr \\
&\quad + \int_{-h}^0 \|rV^{(s)}(h+r)\Phi(s)f\|dr \\
&\leq C(\|x\| + \|f\|_1 + \|f'\|_1) \int_{-h}^0 (h+r)^2 dr + \|\Phi\|_\infty \|f\|_1 \frac{h^2}{2} \\
&= C(\|x\| + \|f\|_1 + \|f'\|_1) h^2.
\end{aligned}$$

The proof is hence complete. \square

We now can prove the first order convergence of the sequential splitting. Passing from local error estimates to convergence is done by the standard trick of telescopic summation.

Theorem 3.3. *For every $T_0 > 0$ there is constant $C > 0$ such that for all $f \in W^{1,1}([-1, 0]; X)$ and $x = f(0)$ the inequality*

$$\left\| \prod_{j=0}^{n-1} \mathcal{T}^{(s+jh)}(h) \mathcal{S}^{(s+jh)}(h) \begin{pmatrix} x \\ f \end{pmatrix} - \mathcal{W}(t, s) \begin{pmatrix} x \\ f \end{pmatrix} \right\| \leq \frac{C(t-s)^2}{n} (\|x\| + \|f\|_1 + \|f'\|_1)$$

holds for all $s \in [-T_0, T_0]$, $t \in [s, s + T_0]$ and for all $n \in \mathbb{N}$, where $h = \frac{t-s}{n}$.

Proof. Take $n_0 \in \mathbb{N}$ so large that $T_0/n_0 < 1$ holds. Fix $n \geq n_0$, t, s as in the assertion and set $h := (t-s)/n$. By telescopic summation we obtain

$$\begin{aligned}
(9) \quad & \prod_{j=0}^{n-1} \mathcal{T}^{(s+jh)}(h) \mathcal{S}^{(s+jh)}(h) \begin{pmatrix} x \\ f \end{pmatrix} - \mathcal{W}(t, s) \begin{pmatrix} x \\ f \end{pmatrix} \\
&= \prod_{j=0}^{n-1} \mathcal{T}^{(s+jh)}(h) \mathcal{S}^{(s+jh)}(h) \begin{pmatrix} x \\ f \end{pmatrix} - \prod_{j=0}^{n-1} \mathcal{W}(s + (j+1)h, s + jh) \begin{pmatrix} x \\ f \end{pmatrix} \\
&= \sum_{k=0}^{n-1} \left(\left(\prod_{j=k+1}^{n-1} \mathcal{T}^{(s+jh)}(h) \mathcal{S}^{(s+jh)}(h) \right) \times \right. \\
&\quad \times [\mathcal{T}^{(s+kh)}(h) \mathcal{S}^{(s+kh)}(h) - \mathcal{W}(s + (k+1)h, s + kh)] \times \\
&\quad \times \left. \left(\prod_{j=0}^{k-1} \mathcal{W}(s + (j+1)h, s + jh) \right) \begin{pmatrix} x \\ f \end{pmatrix} \right) = \\
&= \sum_{k=0}^{n-1} \left(\left(\prod_{j=k+1}^{n-1} \mathcal{T}^{(s+jh)}(h) \mathcal{S}^{(s+jh)}(h) \right) \times \right. \\
&\quad \times [\mathcal{T}^{(s+kh)}(h) \mathcal{S}^{(s+kh)}(h) - \mathcal{W}(s + (k+1)h, s + kh)] \times
\end{aligned}$$

$$\times \mathcal{W}(s + kh, h) \binom{x}{f} \Bigg).$$

For

$$\binom{x_k}{f_k} := \mathcal{W}(s + kh, s) \binom{x}{f}$$

we have $\|x_k\|, \|f_k\| \leq C \|\binom{x}{f}\|$. Therefore can we conclude from Theorem 3.2 that

$$\left\| \left(\mathcal{T}^{(s+kh)}(h) \mathcal{S}^{(s+kh)}(h) - \mathcal{W}(s + (k+1)h, s + kh) \right) \binom{x_k}{f_k} \right\| \leq Ch^2 (\|x\| + \|f\|_1 + \|f'\|_1)$$

holds for all $k = 0, \dots, n-1$. From this and from (9) it follows

$$\left\| \prod_{j=0}^{n-1} \mathcal{T}^{(s+jh)}(h) \mathcal{S}^{(s+jh)}(h) \binom{x}{f} - \mathcal{W}(t, s) \binom{x}{f} \right\| \leq nCh^2 (\|x_k\| + \|f_k\|_1 + \|f'_k\|_1),$$

hence the assertion. \square

4. NUMERICAL EXAMPLES

In this section we present our numerical examples obtained by the numerical code which applies the scheme described in Theorem 2.2 and in (1). The program code we apply is a modification of the code appearing in Csomós and Nickel [5]. In order to check the convergence of the numerical scheme in the nonautonomous case, and compare the solutions in the autonomous and nonautonomous cases, we will investigate the following examples.

Example 4.1. Let $X = \mathbb{R}$, $B = b \in \mathbb{R}$ and consider

$$\begin{cases} \frac{d}{dt}u(t) = bu(t) + \int_{-1}^0 \mu(t, \sigma)u(t + \sigma)d\sigma, & t \geq 0, \\ u(0) = x \in \mathbb{R}, \\ u_0 = f \in L^1([-1, 0]; \mathbb{R}), \end{cases}$$

for some $\mu \in L^\infty(\mathbb{R} \times [-1, 0])$. In this case the delay operator $\Phi(t)$ is defined by

$$\Phi(t)g = \int_{-1}^0 \mu(t, \sigma)g(\sigma)d\sigma$$

for all $g \in L^1([-1, 0]; \mathbb{R})$. Let us choose the initial values as $x = 1$ and $f(\sigma) = 1 - \sigma$ for $\sigma \in [-1, 0]$, and $b = -1$. As a particular example we choose the following functions μ :

- a) $\mu(t, \sigma) = 1$ in the autonomous case, and
 - b) $\mu(t, \sigma) = 1 - \sin t$ in the nonautonomous case,
- for $t \geq 0$ and $\sigma \in [-1, 0]$.

Example 4.2. Let us consider $X = \mathbb{R}$, $B = b \in \mathbb{R}$ and

$$\begin{cases} \frac{d}{dt}u(t) = bu(t) + \mu(t)u(t-1), & t \geq 0, \\ u(0) = x \in \mathbb{R}, \\ u_0 = f \in L^1([-1, 0]; \mathbb{R}). \end{cases}$$

The delay operator in this case is

$$\Phi(t)g = \mu(t)g(-1)$$

for all $g \in W^{1,1}([-1, 0]; \mathbb{R})$. Let choose the initial values again as $x = 1$ and $f(\sigma) = 1 - \sigma$ for $\sigma \in [-1, 0]$, and $b = -1$. As above, we consider the functions μ again:

- a) $\mu(t, \sigma) = 1$ in the autonomous case, and
 b) $\mu(t, \sigma) = 1 - \sin t$ in the nonautonomous case,
 for $t \geq 0$ and $\sigma \in [-1, 0]$.

Convergence of the numerical scheme. In order to examine the convergence of the numerical scheme described in Theorem 2.2, we plot the values of split solution u_n obtained by the formula

$$u_n = \pi_1 \prod_{p=0}^{n-1} \begin{pmatrix} V^{(s+ph)}(h) & 0 \\ V_h^{(s+ph)} & T(h) \end{pmatrix} \begin{pmatrix} I & h\Phi(s+ph) \\ 0 & \tilde{I} \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix}$$

for different values of time step h . The results for the nonautonomous case are shown on Figure 1. One can see that the split solutions converges to the exact solution if h decreases. The convergence of this numerical scheme in the autonomous case has been already investigated in Csomós, Nickel [5, Section 5.3].

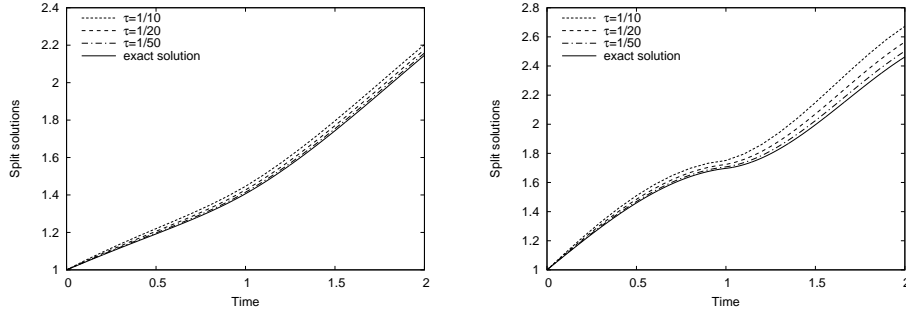


FIGURE 1. Results on the convergence of the numerical scheme for the nonautonomous delay equation with delay functions as in Example 4.1 (left panel) and Example 4.2 (right panel).

Long-time behaviour (difference between autonomous and nonautonomous cases). The long-time behaviour of split solutions u_n of the autonomous and nonautonomous delay equations is shown on Figure 2 for the delay functions in Examples 4.1 (left panel) and 4.2 (right panel). It can be clearly seen that in the case of the nonautonomous equation the difference in the delay functions does not play any qualitative role, because the effect of the function μ (i.e. the sin wave) suppresses it.

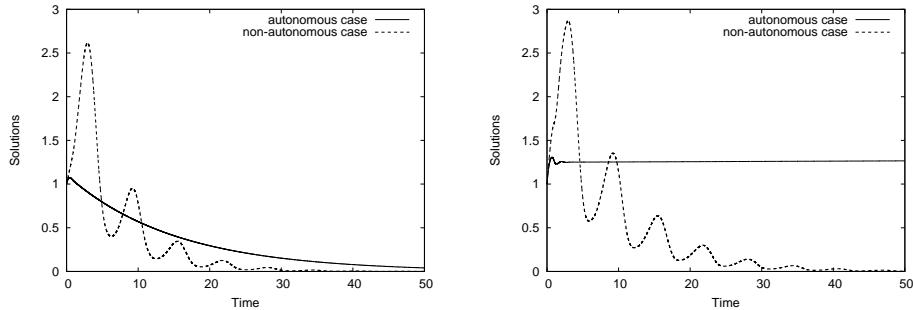


FIGURE 2. Long-time behaviour of split solutions of the autonomous and nonautonomous delay equations with delay functions shown in Example 4.1 (left panel) and Example 4.2 (right panel).

Difference between delay functions in Examples 4.1 and 4.2. On Figure 3 the effect of the different delay functions are shown in the autonomous and nonautonomous cases, respectively. As we have already seen, the effect of the delay function is suppressed by the sin wave of function μ in the nonautonomous case. In the autonomous case, however, the (structure of the) delay function $\Phi(t)$ plays an important role.

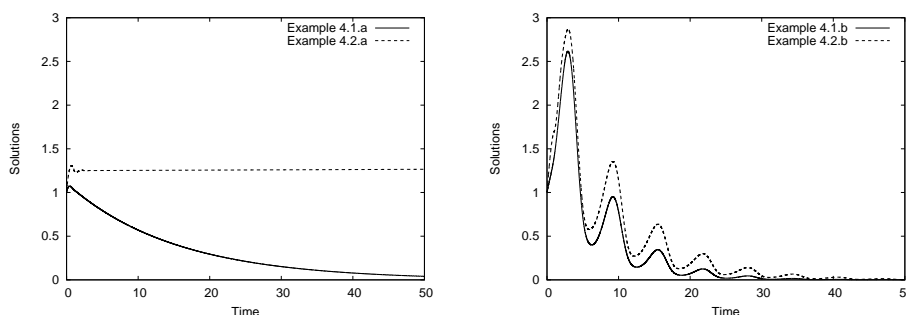


FIGURE 3. Effect of the different delay functions in the autonomous (left panel) and nonautonomous (right panel) cases.

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